Digital Communication Systems
ECS 452

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5.2 Binary Convolutional Codes

Binary Convolutional Codes

- Introduced by Elias in 1955
  - There, it is referred to as convolutional parity-check symbols codes.
  - Peter Elias received
    - Claude E. Shannon Award in 1977
    - IEEE Richard W. Hamming Medal in 2002
      - for "fundamental and pioneering contributions to information theory and its applications
- The encoder **has memory**.
  - In other words, the encoder is a **sequential circuit** or a **finite-state machine**.
    - Easily implemented by shift register(s).
    - The **state** of the encoder is defined as the contents of its memory.

Binary Convolutional Codes

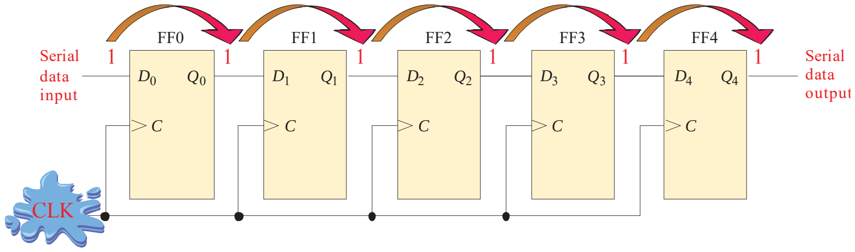
- The encoding is done on a **continuous** running basis rather than by blocks of k data digits.
  - So, we use the terms **bit streams** or **sequences** for the input and output of the encoder.
  - In theory, these sequences have infinite duration.
  - In practice, the state of the convolutional code is periodically forced to a known state and therefore code sequences are produced in a block-wise manner.

Binary Convolutional Codes

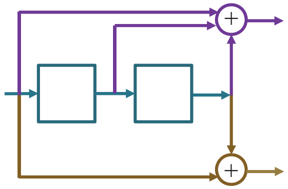
- In general, a **rate-$\frac{k}{n}$ convolutional encoder** has
  - k shift registers, one per input information bit, and
  - n output coded bits that are given by linear combinations (over the binary field, of the contents of the registers and the input information bits.
- k and n are usually small.
- For simplicity of exposition, and for practical purposes, only **rate-$\frac{1}{n}$** binary convolutional codes are considered here.
  - $k = 1$.
  - These are the most widely used binary codes.

(Serial-in/Serial-out) Shift Register

- Accept data serially: one bit at a time on a single line.
- Each clock pulse will move an input bit to the next FF. For example, a 1 is shown as it moves across.
- Example: five-bit serial-in serial-out register.



Example 1: $n = 2$, $k = 1$



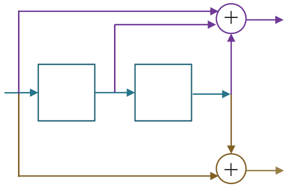
Graphical Representations

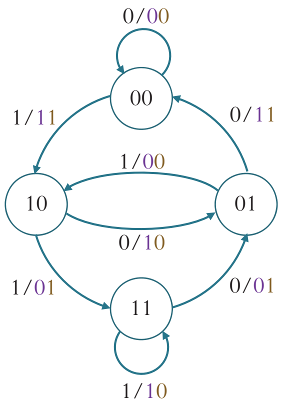
- Three different but related graphical representations have been devised for the study of convolutional encoding:

1. the state diagram
2. the code tree
3. the trellis diagram

Ex. 1: State (Transition) Diagram

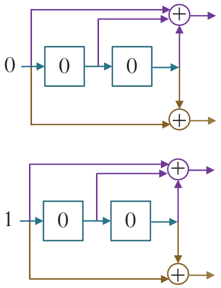
- The encoder behavior can be seen from the perspective of a finite state machine with its state (transition) diagram.

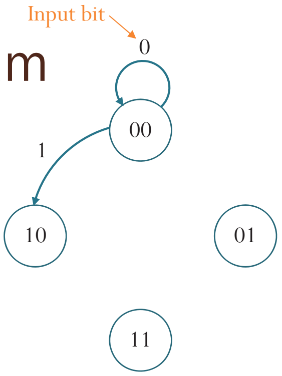


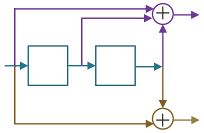


A four-state directed graph that uniquely represents the input-output relation of the encoder.

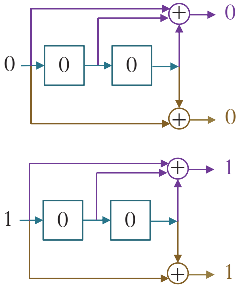
Drawing State Diagram

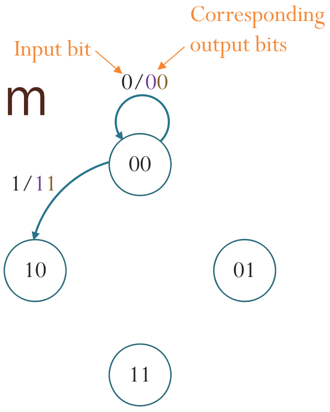


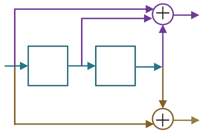




Drawing State Diagram







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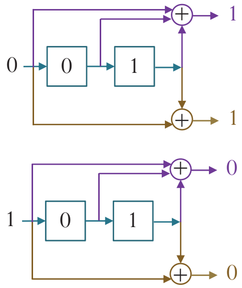
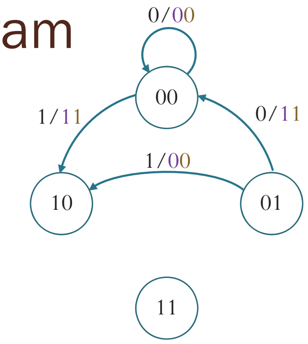
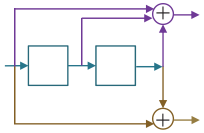
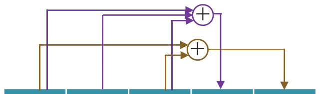


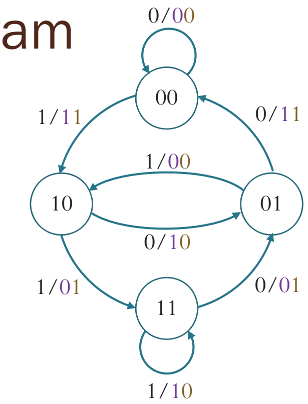


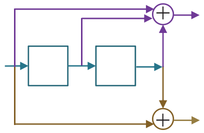


Drawing State Diagram

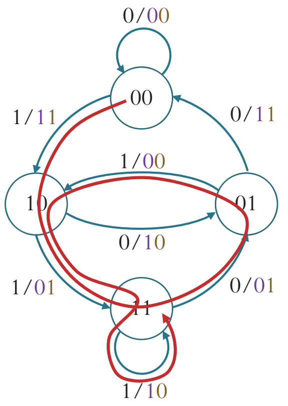


b	s_0	s_1	$x^{(1)}$	$x^{(2)}$
0	0	0	0	0
1	0	0	1	1
0	0	1	1	1
1	0	1	0	0
0	1	0	1	0
1	1	0	0	1
0	1	1	0	1
1	1	1	1	0

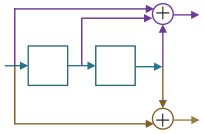




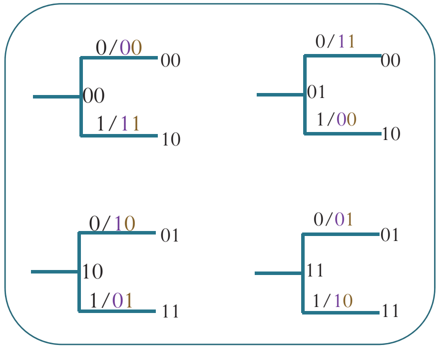
Tracing the State Diagram to Find the Outputs

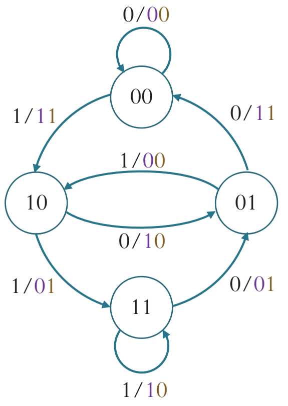


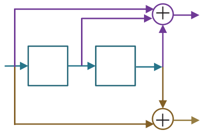
Input	1	1	0	1	1	1
Output	11	01	01	00	01	10



Parts for Code Tree





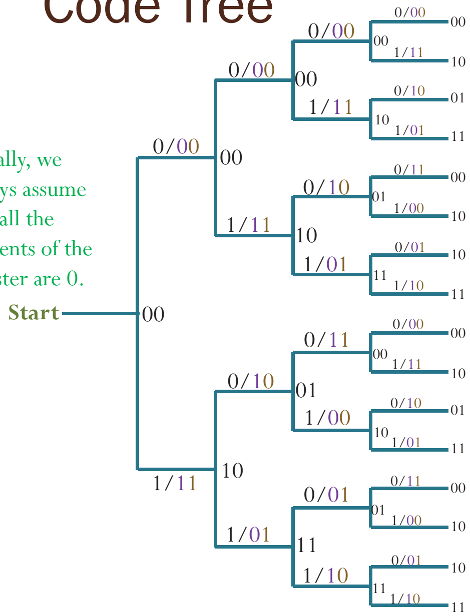


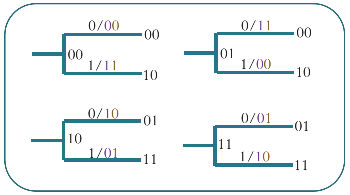
Two branches initiate from each node,
the upper one for 0 and
the lower one for 1.

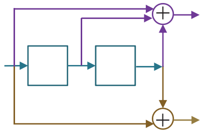
Show the coded output for any possible sequence of data digits.

Code Tree

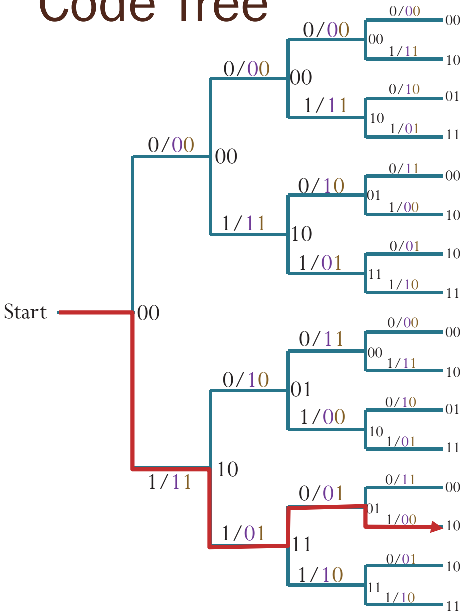
Initially, we always assume that all the contents of the register are 0.



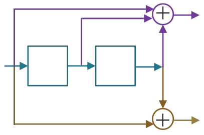




Code Tree

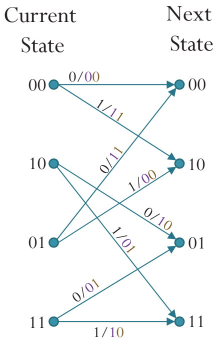


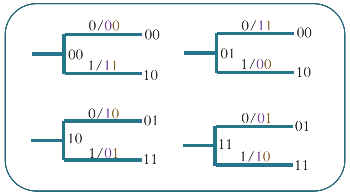
Input	1	1	0	1
Output	11	01	01	00

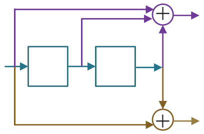


Code Trellis

[Carlson & Crilly, 2009, p. 620]

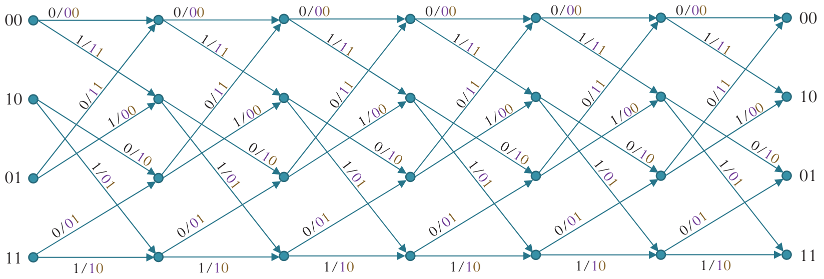


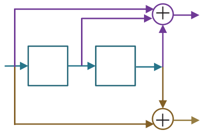




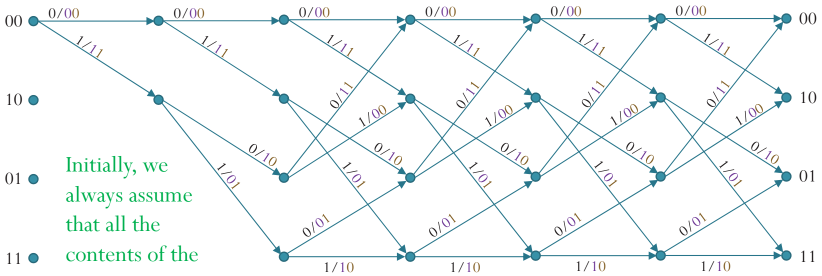
Towards the Trellis Diagram

Another useful way of representing the code tree.



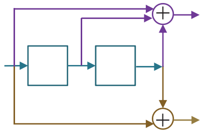


Trellis Diagram



Initially, we always assume that all the contents of the register are 0.

Each path that traverses through the trellis represents a valid codeword.



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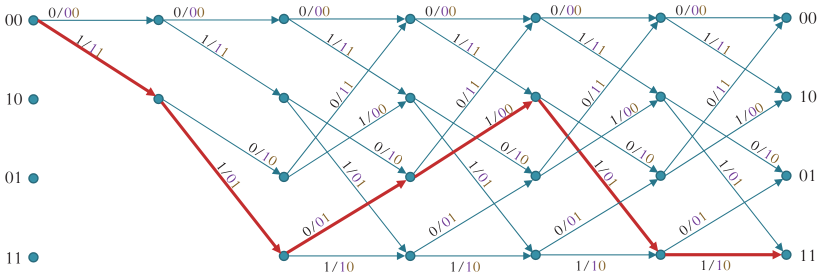
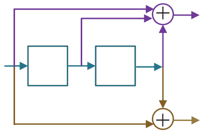
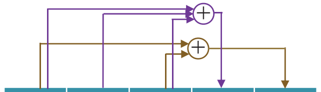


Input	1	1	0	1	1	1
Output	11	01	01	00	01	10

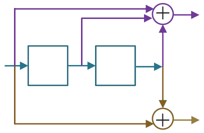


Directly Finding the Output



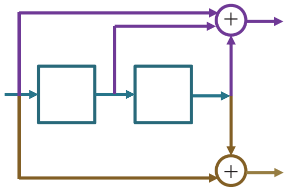
b	s_0	s_1	$x^{(1)}$	$x^{(2)}$
1	0	0		
1				
0				
1				
1				
1				

Input	1	1	0	1	1	1
Output						



Direct Minimum Distance Decoding

- Suppose $\underline{\mathbf{y}}$ = [11 01 11].
- Find $\hat{\underline{\mathbf{b}}}$.
  - Find the message $\hat{\underline{\mathbf{b}}}$ which corresponds to the (valid) codeword $\hat{\underline{\mathbf{x}}}$ with minimum (Hamming) distance from $\underline{\mathbf{y}}$.
    - $\hat{\underline{\mathbf{x}}} = \arg\min_{\underline{\mathbf{x}}} d(\underline{\mathbf{x}}, \underline{\mathbf{y}})$



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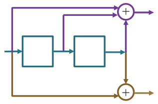

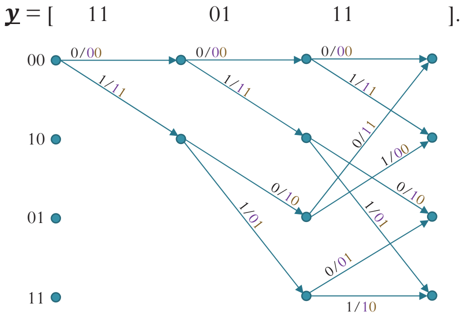

For 3-bit message,
there are $2^3 = 8$ possible codewords.
We can list all possible codewords.
However, here, let's first try to work on the distance directly.

Direct Minimum Distance Decoding

- Suppose $\underline{\mathbf{y}}$ = [11 01 11].
- Find $\hat{\underline{\mathbf{b}}}$.
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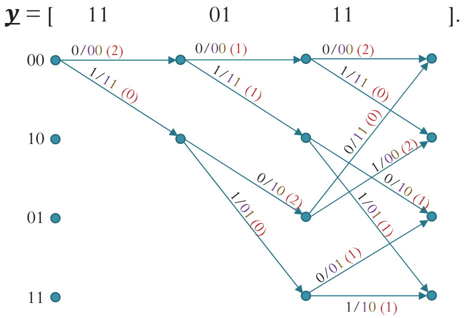
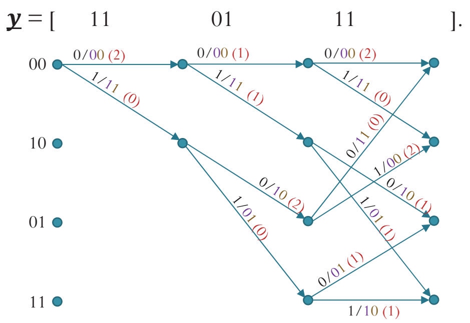


The number in parentheses on each branch is the branch metric, obtained by counting the differences between the encoded bits and the corresponding bits in $\underline{\mathbf{y}}$.

Direct Minimum Distance Decoding

- Suppose $\underline{\boldsymbol{y}}$ = [11 01 11].
- Find $\hat{\underline{\mathbf{b}}}$.
  - Find the message $\hat{\underline{\mathbf{b}}}$ which corresponds to the (valid) codeword $\hat{\underline{\mathbf{x}}}$ with minimum (Hamming) distance from $\underline{\boldsymbol{y}}$.



$\underline{\mathbf{b}}$	$d(\underline{\mathbf{x}},\underline{\mathbf{y}})$
000	2+1+2 = 5
001	2+1+0 = 3
010	2+1+1 = 4
011	2+1+1 = 4
100	0+2+0 = 2
101	0+2+2 = 4
110	0+0+1 = 1
111	0+0+1 = 1

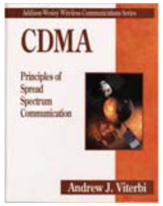
Viterbi decoding

- Developed by Andrew J. Viterbi
  - Also co-founded Qualcomm Inc.
- Published in the paper "Error Bounds for Convolutional Codes and an Asymptotically Optimum Decoding Algorithm", IEEE Transactions on Information Theory, Volume IT-13, pages 260-269, in April, **1967**.

https://en.wikipedia.org/wiki/Andrew_Viterbi

Andrew J. Viterbi

- 1991: Claude E. Shannon Award
- 1952-1957: MIT BS & MS
  - Studied electronics and communications theory under such renowned scholars as Norbert Wiener, Claude Shannon, Bruno Rossi and Roberto Fano.



- 1962: Earned one of the first doctorates in electrical engineering granted at the University of Southern California (USC)
  - Ph.D. dissertation: error correcting codes
- 2004: USC Viterbi School of Engineering

  named in recognition of his $52 million gift

Viterbi Decoding: Ex. 1

- Suppose $\underline{\boldsymbol{y}}$ = [11 01 11].
- Find $\hat{\underline{\mathbf{b}}}$.
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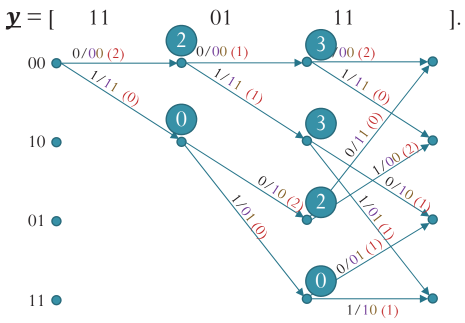
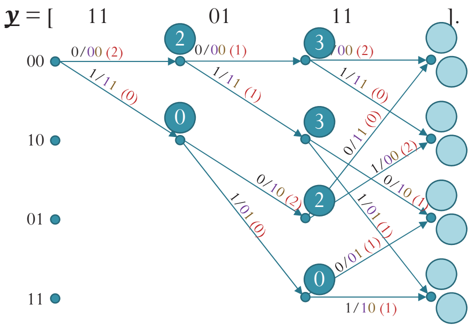


Each **circled number** at a node is the running (cumulative) path metric, obtained by summing branch metrics (distance) up to that node. Here, it is simply the cumulative distance.

Viterbi Decoding: Ex. 1

- Suppose $\underline{\boldsymbol{y}}$ = [11 01 11].
- Find $\hat{\underline{\mathbf{b}}}$.
  - Find the message $\hat{\underline{\mathbf{b}}}$ which corresponds to the (valid) codeword $\hat{\underline{\mathbf{x}}}$ with minimum (Hamming) distance from $\underline{\boldsymbol{y}}$.



- For the last column of nodes, each of the nodes has two branches going into it.
- So, there are two possible cumulative distance values.

Viterbi Decoding: Ex. 1

- Suppose $\underline{\boldsymbol{y}}$ = [11 01 11].
- Find $\hat{\underline{\mathbf{b}}}$.
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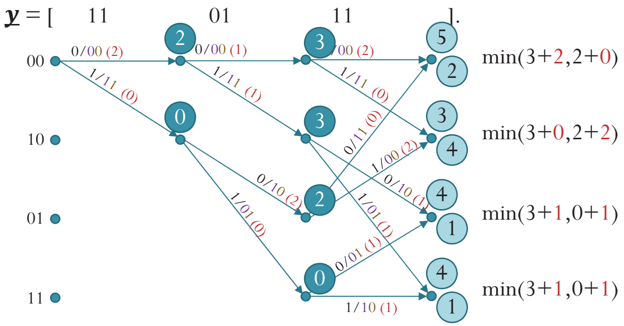

We **discard the larger-distance path** because, regardless of what happens subsequently, this path will have a larger Hamming distance from $\underline{y}$.

Viterbi Decoding: Ex. 1

- Suppose $\underline{\boldsymbol{y}}$ = [11 01 11].
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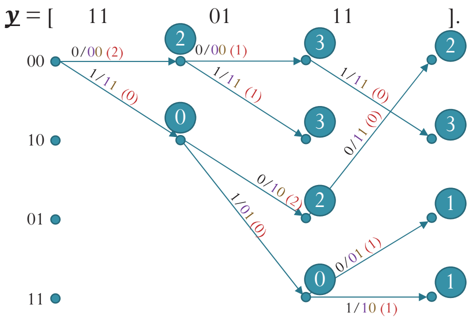

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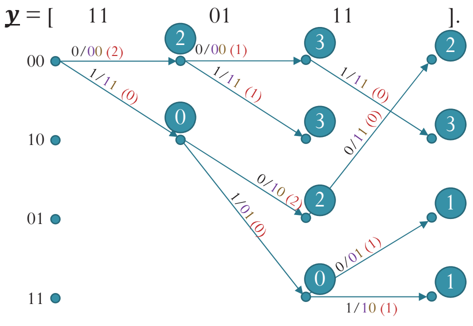
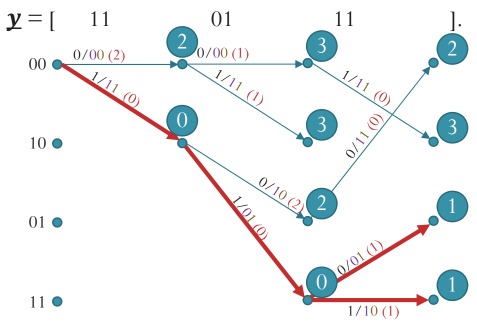


Note that we keep exactly one (optimal) **survivor path** to each state. (Unless there is a tie, then we keep both or choose any.)

Viterbi Decoding: Ex. 1

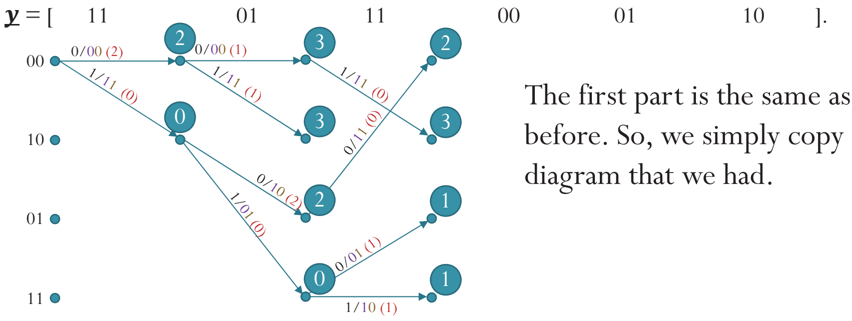
- Suppose $\underline{\boldsymbol{y}} = [11\ 01\ 11]$.
- Find $\hat{\underline{\mathbf{b}}}$.
  - Find the message $\hat{\underline{\mathbf{b}}}$ which corresponds to the (valid) codeword $\hat{\underline{\mathbf{x}}}$ with minimum (Hamming) distance from $\underline{\boldsymbol{y}}$.



- So, the codewords which are nearest to $\underline{\boldsymbol{y}}$ is [11 01 01] or [11 01 10].
- The corresponding messages are [110] or [111], respectively.

Viterbi Decoding: Ex. 2

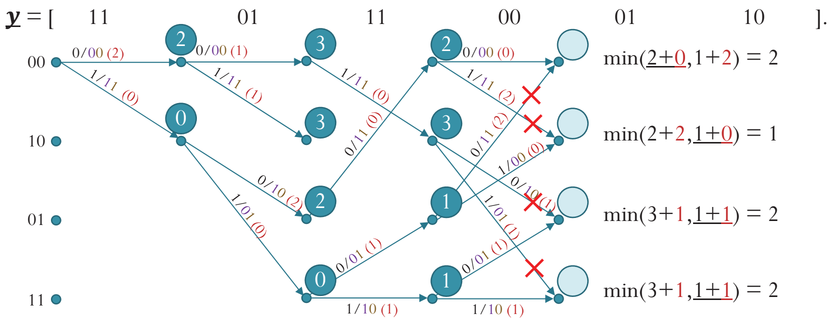
- Suppose $\underline{\boldsymbol{y}} = [\underbrace{11\ 01\ 11}_{\text{same as before}}\ 00\ 01\ 10]$.
- Find $\hat{\underline{\mathbf{b}}}$.



The first part is the same as before. So, we simply copy the diagram that we had.

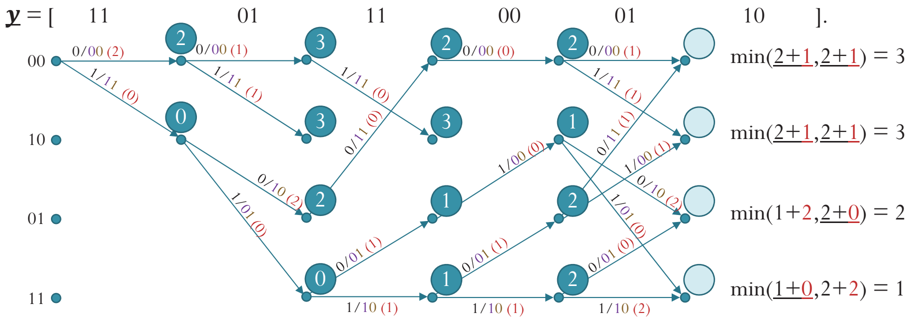
Viterbi Decoding: Ex. 2

- Suppose $\underline{\boldsymbol{y}} = [11\ 01\ 11\ 00\ 01\ 10]$.
- Find $\hat{\underline{\mathbf{b}}}$.



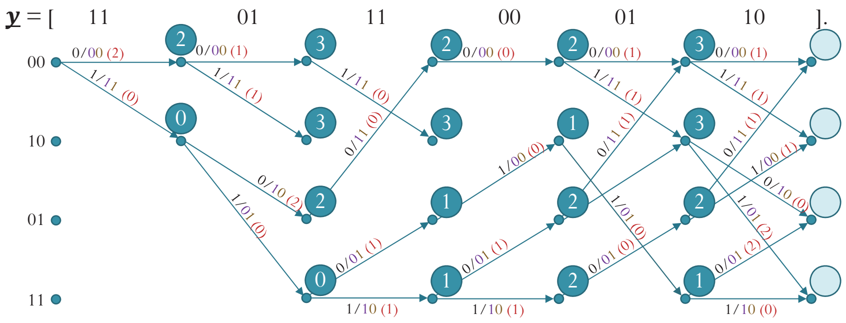
Viterbi Decoding: Ex. 2

- Suppose $\underline{\boldsymbol{y}} = [11\ 01\ 11\ 00\ 01\ 10]$.
- Find $\hat{\underline{\mathbf{b}}}$.



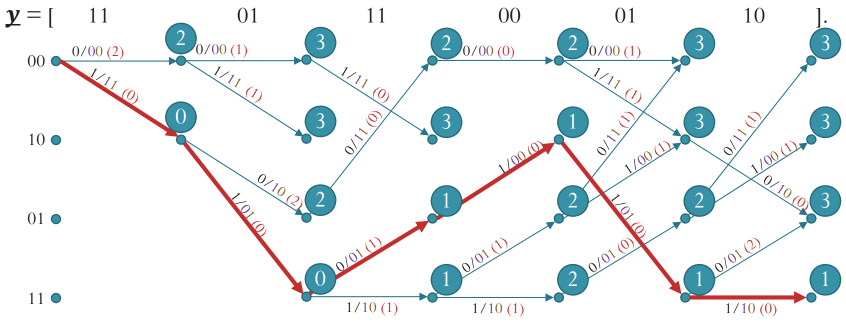
Viterbi Decoding: Ex. 2

- Suppose $\underline{\boldsymbol{y}}$ = [11 01 11 00 01 10].
- Find $\underline{\hat{\mathbf{b}}}$.



Viterbi Decoding: Ex. 2

- Suppose $\underline{\boldsymbol{y}}$ = [11 01 11 00 01 10].
- Find $\underline{\hat{\mathbf{b}}}$.

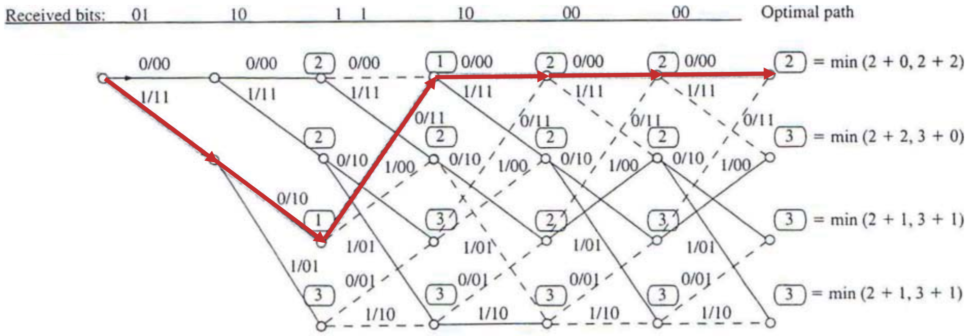


$\underline{\hat{\mathbf{x}}}$ = [11 01 01 00 01 10]

$\underline{\hat{\mathbf{b}}}$ = [1 1 0 1 1 1]

Viterbi Decoding: Ex. 3

- Suppose $\underline{\boldsymbol{y}}$ = [01 10 11 10 00 00].



$\underline{\hat{\mathbf{x}}}$ = [11 10 11 00 00 00]

$\underline{\hat{\mathbf{b}}}$ = [1 0 0 0 0 0]

Reference

- Chapter 15 in [Lathi & Ding, 2009]
- Chapter 13 in [Carlson & Crilly, 2009]
- Section 7.11 in [Cover and Thomas, 2006]